

Complex Treatment of a Class of Nonconservative Stability Problems

by S. N. Prasad and G. Herrmann

No. 17991

FACILITY FORM 602

(ACCESSION NUMBER) 11 27

(PAGES) CR 81622

(NASA CR OR TMX OR AD NUMBER)

(THRU) _____

(CODE) 30

(CATEGORY) _____

National Aeronautics and Space Administration
Grant NsG 605

January 1967
Technical Report No. 67-1

GPO PRICE \$ _____

CFSTI PRICE(S) \$ _____

Hard copy (HC) 3.00

Microfiche (MF) .65

COMPLEX TREATMENT OF A CLASS OF NONCONSERVATIVE
STABILITY PROBLEMS*

by

S. N. Prasad

Research Engineer
Department of Civil Engineering
The Technological Institute
Northwestern University
Evanston, Illinois

and

G. Herrmann

Professor of Civil Engineering
The Technological Institute
Northwestern University
Evanston, Illinois

* This research was supported by the National Aeronautics and Space Administration under Grant NsG 605.

Abstract

This study is concerned with certain aspects of the theory of stability of continuous, one-dimensional, nonconservative elastic systems. In particular, it is pointed out that several types of such systems may be described analytically by the same non-self-adjoint boundary value problem involving a complex differential equation of motion. The imaginary part of the complex force parameter is of interest because it may be associated with a destabilizing effect, the loss of stability being due primarily to the real part of the complex force. The physical origin of the imaginary part may be associated with velocity-dependent forces (such as viscous damping, Coriolis acceleration and other gyroscopic forces), and certain other forces which need not be velocity-dependent, e.g., a torque which remains in the plane of deformation of a cantilevered bar.

Several general theorems applicable to the class of systems selected for study are established and an example serves to illustrate various features of this class.

Introduction

An important group of problems in the field of applied mechanics deals with the stability analysis of elastic systems subjected to nonconservative forces. In this group, nonconservative forces are defined as being nondissipative and not derivable from a potential. Tangential (or "follower") forces are examples of loads that are nonconservative; such forces have also been termed circulatory by Ziegler [1,2]*. It is known by now that a direct procedure for stability analysis of such systems consists of obtaining the variational equations of Poincaré by assuming infinitesimal deviations from the undisturbed equilibrium position, and in studying then the subsequent motion [3]. The forces which act on the system may or may not depend on time explicitly. In the following, however, they are taken to be independent of time (autonomous system). Of course they will be functions of deformations and time rate of change of displacements of the system. Therefore, when a dependence $e^{i\omega t}$ in time is assumed, attention is focused on determining the property of the frequency ω .

Small, free, undamped oscillations of spatially one-dimensional conservative systems are governed by self-adjoint partial differential equations of motion, so that the frequencies ω_j , $j = 1, 2, \dots, \infty$, are all real. When nonconservative forces of the type just mentioned are present, however, the governing equation of motion is non-self-adjoint so that the solutions for ω_j may be complex. Thus, the location of ω_j in the complex

*Numbers in brackets designate References at end of paper.

$i\omega$ -plane will determine stability of such systems. This criterion forms the basis of the analysis presented in the sequel.

That group of problems in which the forces (or moments) are idealized to be of the follower type is, mathematically, relatively simple. It is recognized, however, that in many problems the analysis will have to include terms originating from Coriolis acceleration or from other gyroscopic effects. Such terms may be recognized as being velocity-dependent forces, and in the equations describing small oscillations for stability analysis they occur as mixed time and space derivatives (the order of the former is generally odd). As a consequence, the resulting ordinary differential equation (in spatial coordinate) will have terms whose coefficients are imaginary. For instance, the governing equations for a cantilevered elastic pipe conveying fluid possess such a feature, and this problem was considered in [4].

The stability analysis of elastic systems moving relatively to a surrounding fluid is another instance when one has to deal with differential equations that have terms with complex coefficients. In this case the expressions for the forces exerted by the fluid on the elastic system are not completely specified a priori; rather, they are determined in the course of solving the problem. Examples belonging to this class include the flutter analysis of hydrofoils, airplane wings, and elastic panels [3,5-9].

Further, it is often necessary to consider internal damping in the material. One way to account for such damping is to select some suitable form of a linear viscoelastic stress-strain relationship (a common form of

which may consist of a certain combination of linear springs and dashpots such that the relation is obtained in the form of linear differential operators that act on stress and strain [10]. The terms in the equation of motion that arise due to the strain energy of the elastic system, therefore, change so that they have complex coefficients when dependence on time is assumed to be of the form $e^{i\omega t}$, as discussed earlier. Such considerations, which give rise to several interesting phenomena, were introduced in the studies referred in [4,9] by using viscoelastic material, a common form of which is given by the Kelvin model.

Finally, there are certain position-dependent (but velocity-independent) forces (e.g., a torque which remains in the plane of deformation of a cantilevered bar as in Nikolai's problem [11]) which also give rise to complex terms in the differential equation of motion, in a manner similar to velocity-dependent forces, as discussed before.

It is, therefore, desirable to study in a general way some of the properties of the stability of elastic systems subjected to such "complex" forces. The term "complex" is used here for want of a better designation of the class of forces in question, and it is stressed again that their physical origin can be quite diverse. Yet, since their mathematical description can be made identical, they obviously must possess some common properties. In particular, as will be shown in the sequel, a sufficiently small imaginary part of the complex force parameter is associated with a destabilizing effect, the loss of stability being due primarily to the real part of the complex force. The destabilizing effect of viscous damping on the stability of a linear, discrete, nonconservative system was

first discovered by Ziegler [1,2], and later several authors [3,4,9,12-15] explored this interesting phenomenon in more detail. In the present study several general theorems concerning the destabilizing effects associated with small imaginary part of complex force are established, and by way of an example a more general form of Nikolai's problem is discussed in the perspective of these results.

Some General Properties of "Complex" Forces

Consider the following form of an ordinary linear differential equation:

$$P(u) + (iv)Q(u) = \omega^2 f(x)u ; \quad i = (-1)^{\frac{1}{2}} \quad (1)$$

where v is a positive parameter which denotes the order of magnitude, u is a complex function of a real variable x , ω is the circular frequency of small oscillation, and $f(x)$ is a known continuous function of x ; and

$$P(u) = \sum_{n=1}^N \alpha_n(x) \frac{d^n u}{dx^n} \quad (2)$$

$$Q(u) = \sum_{n=1}^N \beta_n(x) \frac{d^n u}{dx^n}$$

At the end points of the interval $a \leq x \leq b$ in which (1) is valid, the N boundary conditions are

$$L_n(u) = 0 , \quad \text{at } x = a ; \quad n = 0, 1, \dots, r$$

$$L_n(u) = 0 , \quad \text{at } x = b ; \quad n = r+1, r+2, \dots, N-1 \quad (3)$$

where the expressions $L_n(u)$ are of the form

$$L_j(u) = \sum_{n=0}^{N-1} \left\{ \eta_{jn} + (iv)\theta_{jn} \right\} \frac{d^n u}{dx^n} ; \quad j = 0, 1, \dots, N-1 \quad (4)$$

η_{jn} and θ_{jn} are real quantities characterizing certain properties (such as stiffness or inertia) at the two end points.

It will be assumed that the coefficients $f(x)$, $\alpha_n(x)$, and $\beta_n(x)$, are continuous, single-valued, real functions of x throughout the interval $a \leq x \leq b$, and α_N does not vanish at any point of the interval. Also, some of α_n , β_n , η_{jn} and θ_{jn} will be assumed single-valued continuous functions of F , which denotes the magnitude of loading and is a real positive parameter. Eq. (1) has at most N linearly independent solutions $U_n(x)$; $n = 1, 2, \dots, N$, which will be continuous functions of the coefficients $(\alpha_n + i\nu\beta_n)$ and ω . The dependence of $U_n(x)$ on $\alpha_n(x)$ and $\beta_n(x)$ will be indicated merely by parameters α_n and β_n , respectively. As mentioned above, β_n stands for the imaginary part of the complex forces. Then, the most general solution u can be written in a functional form as

$$u = \sum_{m=1}^N A_m U_m(x, (\alpha_n + i\nu\beta_n), \omega) ; \quad n = 1, 2, \dots, N \quad (5)$$

where A_m are N arbitrary complex constants.

Substituting the solution given by Eq. (5) into the expressions for boundary conditions (3), a set of N linear, homogeneous, algebraic equations in A_m is obtained, the coefficients of which are functions of $(\alpha_n + i\nu\beta_n)$, ω , and $(\eta_{jn} + i\nu\theta_{jn})$. A nontrivial solution of A_m exists if and only if the determinant, denoted by $\bar{\Delta}$, of the coefficients of A_m is zero, which is expressed in a functional form as follows:

$$\bar{\Delta} \equiv \bar{\Delta}((\alpha_n + i\nu\beta_n), \omega, (\eta_{jk} + i\nu\theta_{jk})) = 0 ; \quad \begin{matrix} j, k = 0, 1, \dots, N-1 \\ n = 1, 2, \dots, N \end{matrix} \quad (6)$$

It is noted that the operator $(P + i\nu Q)$, with the boundary conditions

(3), does not have a singularity for $\nu = 0$ and has continuous dependence on ν . Hence, by setting $\nu = 0$ in the function $\bar{\Delta}$, another function denoted by $\Delta(\alpha_n, \omega, \eta_{jk})$ is obtained which corresponds to the case when the imaginary parts of complex forces vanish. The heart of stability analysis of nonconservative systems in the absence of imaginary parts of complex forces lies in studying the roots of Δ . This will be explained briefly below.

The parameter F represents terms in the operator P and boundary conditions (3) which are caused by external loadings. When F is identically zero, the problem reduces to finding the natural frequencies of the elastic system. For this case the operator P , with respect to boundary conditions that relate to free oscillation, becomes self-adjoint, such that the eigenfrequencies ω_j ; $j = 1, 2, \dots, \infty$, corresponding to the zeros of

$$\Delta(\alpha_n, \omega, \eta_{jk})|_{F=0} = 0 ; \quad j, k = 0, 1, \dots, N-1$$

$$n = 1, 2, \dots, N \quad (7)$$

are all real. It will be assumed that they are distinct and that they remain nonzero in order to eliminate divergence. If F is sufficiently small, these roots ω_j of $\Delta = 0$ are still nonzero and real. As F is increased, at least two of ω_j approach each other and at a certain value of F they coalesce. This is, obviously, a very stringent requirement on the character of the external loadings and is mathematically known to exist for several problems in this field [3]. If F is increased beyond this value, at least two of the roots of the above equation are nonzero, complex conjugates of each other. Hence an unstable state exists in which the amplitudes of oscillations increase exponentially (flutter). The

minimum value of F , say F_e , for which only one pair of real ω satisfies simultaneously the following equations, will correspond to flutter:

$$\begin{aligned}\Delta(\alpha_n, \omega, \eta_{jk}) &= 0 \\ \frac{\partial \Delta}{\partial \omega}(\alpha_n, \omega, \eta_{jk}) &= 0\end{aligned}\tag{8}$$

We return now to the case of nonzero v . At this point it is important to investigate the dependence of β_n and θ_{jk} on ω . It may be pointed out that in case of velocity-dependent forces these terms are associated with odd time derivatives so that $\beta_n(-\omega) = -\beta_n(\omega)$ and $\theta_{jk}(-\omega) = -\theta_{jk}(\omega)$. On the other hand, the following theorem is proved below:

Theorem I. If each β_n and θ_{jk} is a symmetric function of ω (or independent of ω) such that $\beta_n(-\omega) = \beta_n(\omega)$ and $\theta_{jk}(-\omega) = \theta_{jk}(\omega)$, then the system is unstable for any nonvanishing magnitude of the load parameter F .

Proof. Let $\omega = \alpha + iv\beta$ and expand $\bar{\Delta}$ from (6) in powers of v to obtain

$$\begin{aligned}\bar{\Delta}(\alpha + iv\beta, (\alpha_n + iv\beta_n), (\eta_{jk} + iv\theta_{jk})) &= \Delta(\alpha, \alpha_n, \eta_{jk}) + (iv) \left[\frac{\partial \Delta}{\partial \alpha} \beta \right. \\ &+ \sum_{n=1}^N \frac{\partial \Delta}{\partial \alpha_n} \beta_n + \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \frac{\partial \Delta}{\partial \eta_{jk}} \theta_{jk} \left. \right] + \frac{(iv)^2}{2} \left[\frac{\partial^2 \Delta}{\partial \alpha^2} \beta^2 + \sum_{m=1}^N \sum_{n=1}^N \frac{\partial^2 \Delta}{\partial \alpha_m \partial \alpha_n} \beta_m \beta_n \right. \\ &+ 2 \sum_{n=1}^N \frac{\partial^2 \Delta}{\partial \alpha_n \partial \alpha} \beta_n \beta + \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \frac{\partial^2 \Delta}{\partial \eta_{mn} \partial \eta_{jk}} \theta_{mn} \theta_{jk} \\ &+ 2 \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \frac{\partial^2 \Delta}{\partial \eta_{jk} \partial \alpha} \theta_{jk} \beta + 2 \sum_{n=1}^N \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \frac{\partial^2 \Delta}{\partial \alpha_n \partial \eta_{jk}} \beta_n \theta_{jk} \left. \right] \\ &+ O(v^3) + \dots = 0\end{aligned}\tag{9}$$

If we retain orders of v up to $O(v)$, then the proof of the stated theorem

follows immediately. In fact it is meaningful only to show that the system is unstable for vanishing F . Therefore, neglecting $O(v^2)$ and higher, the following is obtained:

$$\Delta(\alpha, \alpha_n, \eta_{jk}) + (iv) \left[\frac{\partial \Delta}{\partial \alpha} \beta + \sum_{n=1}^N \frac{\partial \Delta}{\partial \alpha_n} \beta_n + \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \frac{\partial \Delta}{\partial \eta_{jk}} \theta_{jk} \right] = 0 \quad (10)$$

Separating the real and imaginary parts and equating them to zero, the following results:

$$\Delta(\alpha, \alpha_n, \eta_{jk}) = 0 \quad (11a)$$

and

$$\frac{\partial \Delta}{\partial \alpha} \beta + \sum_{n=1}^N \frac{\partial \Delta}{\partial \alpha_n} \beta_n + \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \frac{\partial \Delta}{\partial \eta_{jk}} \theta_{jk} = 0$$

so that

$$\beta = - \frac{\sum_{n=1}^N \frac{\partial \Delta}{\partial \alpha_n} \beta_n + \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \frac{\partial \Delta}{\partial \eta_{jk}} \theta_{jk}}{\frac{\partial \Delta}{\partial \alpha}} \quad (11b)$$

It has been remarked before that Δ represents the frequency equation of the systems (1) and (3) for the special case of $v = 0$. Under such a circumstance, then, the systems (1) and (3) will give rise to the same solution $\Delta(\omega, \alpha_n, \eta_{jk}) = \Delta(-\omega, \alpha_n, \eta_{jk})$ with respect to $e^{i\omega t}$ or $e^{-i\omega t}$ because every time ω occurs, it is associated with inertia forces, so that the derivatives with respect to time t are even. Therefore, the coefficients of β_n and θ_{jk} , as well as the denominator $\partial \Delta / \partial \alpha$ in Eq. (11b), are symmetric functions of ω ; and if the requirement of the stated theorem is fulfilled, then β obtained from (11b) is sign invariant (let β be positive) with respect to $\pm \omega$. As a

result the solution for frequency ω takes the following form:

$$\pm\omega = \pm(\alpha + i\nu\beta) \quad (12)$$

where α is the solution of (11a) and β is obtained from (11b) after substitution of α . Relation (12) is, obviously, true for any $0 < F \leq F_e$. Consequently, it is not difficult to realize that one of the solutions $e^{\pm i\omega t}$ will correspond to oscillatory motion with exponentially increasing amplitude (flutter) for any nonvanishing magnitude of F . This concludes the proof.

Nikolai's problem is an example of a physical system that illustrates the above theorem. It may, however, be mentioned that in contrast to the requirement of the statement in Theorem I, there may be a case when not all of β_n and θ_{jk} are symmetric functions ω . Then the critical value of F will be determined by investigating the sign of β given by (11b), together with the roots of (11a); and in general there will be a definite nonzero value of critical F . If in Nikolai's problem viscous damping forces are included, then all the features mentioned before are exhibited. This illuminating case will be discussed by way of an example at the end of the present study.

What has been presented above is a sufficient condition on the character of the imaginary parts of complex forces such that they are self-exciting, i.e., they have in a certain sense the property of negative damping. In the following, however, it will be assumed that no β_n or θ_{jk} is a symmetric function in the sense mentioned. Therefore, when F vanishes it will be assumed that all the positive roots ω_j of Eq. (6) are located in the left half of the imaginary axis in the complex $i\omega$ -plane. As F is increased at least one of these roots ω_j approaches the imaginary axis and for a certain value of F Eq. (6) yields one real root, say ω_c . If F is increased beyond

this critical value, one of the roots of (6) becomes complex with negative imaginary part and, therefore, the amplitude of oscillation increases exponentially with time (flutter). Consequently, the minimum value of F , say F_d , which admits a nonzero positive real root ω_c of (6), corresponds to flutter. With these preliminaries the following theorems can be established:

Theorem II. Let v be a sufficiently small number, such that terms of $O(v^2)$ and higher can be neglected in comparison with those of $O(v)$, then

$$F_d \leq F_e$$

Proof. Let $\omega = \alpha + iv\beta$ and obtain as before the following (neglecting $O(v^2)$ and higher):

$$\Delta(\alpha, \alpha_n, \eta_{jk}) = 0 \quad (13a)$$

$$\beta = - \frac{\sum_{n=1}^N \frac{\partial \Delta}{\partial \alpha_n} \beta_n + \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \frac{\partial \Delta}{\partial \eta_{jk}} \theta_{jk}}{\frac{\partial \Delta}{\partial \alpha}} \quad (13b)$$

It should be noted that for $F = 0$, β is a positive quantity, so that when $t \rightarrow \infty$ the system returns to its equilibrium state.

According to what has been assumed before, namely, that Δ and $\partial \Delta / \partial \alpha$ do not vanish simultaneously for $F < F_e$, it follows that $\partial \Delta / \partial \alpha$ does not change sign in this range of F , such that α is a positive quantity that satisfies (13a). Therefore, ω can be real only if the numerator in (13b), namely,

$$\sum_{n=1}^N \frac{\partial \Delta}{\partial \alpha_n} \beta_n + \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \frac{\partial \Delta}{\partial \eta_{jk}} \theta_{jk}, \text{ vanishes. Hence, } F_d = F < F_e \text{ will cause}$$

flutter only if the following equations are satisfied simultaneously:

$$\Delta(\alpha, \alpha_n, \eta_{jk}) = 0 \quad (13a)$$

$$\sum_{n=1}^N \frac{\partial \Delta}{\partial \alpha_n} \beta_n + \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \frac{\partial \Delta}{\partial \eta_{jk}} \theta_{jk} = 0 \quad (13c)$$

Now, assume that for $F \leq F_e$, the expression (13b) does not vanish.

But, for $F > F_e$ at least two roots of Eq. (13a) are conjugate complex of each other, say $\alpha' \pm i\beta'$, and the denominator of the expression (13b), namely, $\partial \Delta / \partial \alpha$, changes sign, so that β , for this value of $\alpha' \pm i\beta'$, is a negative quantity. Therefore, when $F > F_e$, the following expression for ω is obtained:

$$\omega_{1,2} = \alpha' \pm i\beta' + i\nu\beta \quad (14)$$

where β is a negative quantity and is given by

$$\beta = - \frac{\sum_{n=1}^N \frac{\partial \Delta}{\partial \alpha_n} \beta_n + \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \frac{\partial \Delta}{\partial \eta_{jk}} \theta_{jk}}{\frac{\partial \Delta}{\partial \alpha}} \bigg|_{\alpha = \alpha' \pm i\beta'} \quad (15)$$

Thus $F_d = F > F_e$ is associated with an amplitude which increases exponentially with time (flutter).

In the above it has been shown that when the inequality $F > F_e$ is satisfied, the system loses stability by flutter. The case of equality, namely, $F_d = F = F_e$, was not discussed. At this value of $F = F_e$, $\partial \Delta / \partial \alpha$ vanishes for the double root of (13a). However, if (13c) is satisfied at the same time, the criterion that the boundary of stability is determined by a real root of (6) is fulfilled. The condition of equality is met only in this case. Therefore,

Corollary. If the term $\left[\sum_{n=1}^N \frac{\partial \Delta}{\partial \alpha_n} \beta_n + \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \frac{\partial \Delta}{\partial \eta_{jk}} \theta_{jk} \right]$ is

proportional to $\partial\Delta/\partial\alpha$, the destabilizing effect of sufficiently small complex forces is eliminated.

Note that if (13c) does not vanish together with $\partial\Delta/\partial\alpha$, β becomes indeterminate and the method of retaining orders only up to $O(v)$ seems to be inadequate. Higher order terms must then be retained in the analysis.

Now, retaining orders up to $O(v^2)$ in the expansion (9), the following theorem is proved:

Theorem III. Let v be finite such that $O(v^3)$ and higher can be neglected in comparison with $O(v^2)$ and lower. Then

$$F_d \leq F_e$$

if the following relation holds:

$$\begin{aligned} \sum_{m=1}^N \sum_{n=1}^N \frac{\partial^2 \Delta}{\partial \alpha_m \partial \alpha_n} \beta_m \beta_n + \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \frac{\partial^2 \Delta}{\partial \eta_{mn} \partial \eta_{jk}} \theta_{mn} \theta_{jk} \\ + 2 \sum_{n=1}^N \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \frac{\partial^2 \Delta}{\partial \alpha_n \partial \eta_{jk}} \beta_n \theta_{jk} = \frac{\partial^2 \Delta}{\partial \alpha^2} \beta^2 \end{aligned}$$

Proof. Neglecting $O(v^3)$ and higher in (9), and equating real and imaginary parts, separately, to zero, the following is obtained:

$$\begin{aligned} \Delta(\alpha, \alpha_n, \eta_{jk}) - \frac{v^2}{2} \frac{\partial^2 \Delta}{\partial \alpha^2} \beta^2 - v^2 \sum_{n=1}^N \frac{\partial^2 \Delta}{\partial \alpha_n \partial \alpha} \beta_n \beta - v^2 \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \frac{\partial^2 \Delta}{\partial \eta_{jk} \partial \alpha} \theta_{jk} \beta \\ - \frac{v^2}{2} \left[\sum_{m=1}^N \sum_{n=1}^N \frac{\partial^2 \Delta}{\partial \alpha_m \partial \alpha_n} \beta_m \beta_n + \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \frac{\partial^2 \Delta}{\partial \eta_{mn} \partial \eta_{jk}} \theta_{mn} \theta_{jk} \right. \\ \left. + 2 \sum_{n=1}^N \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \frac{\partial^2 \Delta}{\partial \alpha_n \partial \eta_{jk}} \beta_n \theta_{jk} \right] = 0 \end{aligned} \quad (16)$$

and

$$\beta = - \frac{\sum_{n=1}^N \frac{\partial \Delta}{\partial \alpha_n} \beta_n + \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \frac{\partial \Delta}{\partial \eta_{jk}} \theta_{jk}}{\frac{\partial \Delta}{\partial \alpha}} \quad (13b)$$

Replacing the term in brackets in Eq. (16) by $(\partial^2 \Delta / \partial \alpha^2) \beta^2$, as assumed above, the following results:

$$\Delta(\alpha_n, \alpha, \eta_{jk}) - v^2 \frac{\partial^2 \Delta}{\partial \alpha^2} \beta^2 - v^2 \beta \left[\frac{\partial}{\partial \alpha} \left(\sum_{n=1}^N \frac{\partial \Delta}{\partial \alpha_n} \beta_n + \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \frac{\partial \Delta}{\partial \eta_{jk}} \theta_{jk} \right) \right] = 0$$

Using (13b), the above equation reduces to

$$\Delta(\alpha, \alpha_n, \eta_{jk}) - v^2 \frac{\partial^2 \Delta}{\partial \alpha^2} \beta^2 - v^2 \beta \left[\frac{\partial}{\partial \alpha} \left(-\beta \frac{\partial \Delta}{\partial \alpha} \right) \right] = 0$$

Therefore,

$$\Delta(\alpha, \alpha_n, \eta_{jk}) = 0$$

which is identical to Eq. (13a). Thus, by virtue of the relation assumed in Theorem III, for finite v , Eqs. (13b) and (16) represent exactly the conditions discussed in Theorem II, and hence the proof follows as before.

An important question in this context arises: which of the two types of paths shown in Fig. 1 is followed by the solution of Eq. (6)? Path A denotes stability, whereas B is destabilizing with an increase in v . For finite v , so that orders up to $O(v^2)$ are considered, Eqs. (13c) and (16) can be solved simultaneously to give a relation of the type $F = F(v)$. Then, the answer is given by studying the sign of $d^2 F / dv^2$. Obviously, Theorem III signifies transition from path A to B, or vice versa. Results similar to B have been obtained in analyzing the bending-torsional flutter of a swept wing reported in [9]. It has been found that an increase in the magnitude

of the parameter associated with damping in shear results in a decrease of the flutter parameter denoting speed.

Example -- Stability of a Cantilevered Bar Subjected
to a Partially Following Twisting Moment

Consider small motion of a uniform cantilevered bar, shown in Fig. 2, whose moment of inertia I is the same about any axis. The stress-strain relation of the material is expressed as

$$\sigma = E\epsilon + \eta \frac{\partial \epsilon}{\partial t}$$

where E is the modulus of elasticity and η is the coefficient of viscosity. Using complex deflection $w = u + iv$, where u and v are displacements along x and y axes (Fig. 2), respectively, the equation of motion of the bar subjected to a twisting moment which remains in the plane of deformation and partially follows the slope of the end during motion, is given by

$$EI \frac{\partial^4 w}{\partial z^4} + \eta I \frac{\partial^5 w}{\partial z^4 \partial t} - iL \frac{\partial^3 w}{\partial z^3} + m \frac{\partial^2 w}{\partial t^2} = 0 \quad (17)$$

with the following boundary conditions:

$$w = \frac{\partial w}{\partial z} = 0, \quad \text{at } z = 0$$

$$EI \frac{\partial^2 w}{\partial z^2} + \eta I \frac{\partial^3 w}{\partial z^2 \partial t} - i(1 - \kappa)L \frac{\partial w}{\partial z} = 0, \quad \text{at } z = \ell \quad (18)$$

$$EI \frac{\partial^3 w}{\partial z^3} + \eta I \frac{\partial^4 w}{\partial z^3 \partial t} - iL \frac{\partial^2 w}{\partial z^2} = 0, \quad \text{at } z = \ell$$

where L is the twisting moment which always lies in the plane of deformation and rotates by an angle which is κ times the angle of rotation of the free-end section, m is the mass of the bar per unit length, and ℓ is the length

of the bar.* To consider the effect of external damping, a term $k(\partial w/\partial t)$ may be added to Eq. (17). Introducing the following dimensionless parameters,

$$\begin{aligned} \xi &= \frac{z}{\ell}, & \tau &= t \left(\frac{EI}{m\ell^4} \right)^{\frac{1}{2}}, & M' &= \frac{L\ell}{EI}, \\ \delta' &= \left(\frac{\eta^2 I}{mE\ell^4} \right)^{\frac{1}{2}}, & \gamma' &= \left(\frac{k^2 \ell^4}{mEI} \right)^{\frac{1}{2}}, \end{aligned} \quad (19)$$

Eqs. (17) and (18) take the forms

$$\frac{\partial^4 w}{\partial \xi^4} + \delta' \frac{\partial^5 w}{\partial \xi^4 \partial \tau} - iM' \frac{\partial^3 w}{\partial \xi^3} + \gamma' \frac{\partial w}{\partial \tau} + \frac{\partial^2 w}{\partial \tau^2} = 0 \quad (20)$$

$$w = \frac{\partial w}{\partial \xi} = 0, \quad \text{at } \xi = 0$$

$$\frac{\partial^2 w}{\partial \xi^2} + \delta' \frac{\partial^3 w}{\partial \xi^2 \partial \tau} - iM'(1 - \kappa) \frac{\partial w}{\partial \xi} = 0, \quad \text{at } \xi = 1 \quad (21)$$

$$\frac{\partial^3 w}{\partial \xi^3} + \delta' \frac{\partial^4 w}{\partial \xi^3 \partial \tau} - iM' \frac{\partial^2 w}{\partial \xi^2} = 0, \quad \text{at } \xi = 1$$

To investigate stability as affected by small force parameters, the following notation is introduced:

$$\delta' = \nu \delta, \quad M' = \nu M, \quad \text{and} \quad \gamma' = \nu \gamma,$$

in which ν denotes the order of magnitude (as discussed before), and δ , F and γ are of the order of unity. The solution of Eq. (20) is taken in the form $\psi(\xi)e^{\pm i\omega\tau}$; the stability of the bar will be determined by the behavior of ω . The nature of ω will depend upon the nontrivial solutions of the

*The result for the special case of $\kappa = 0$, namely, that the elastic cantilevered bar is unstable for any nonvanishing magnitude of the twisting moment, was presented by Trösch [17]. See also Ref. [3].

following equation in $\psi(\xi)$:

$$\frac{d^4\psi}{d\xi^4} \pm i\nu\omega\delta \frac{d^4\psi}{d\xi^4} - i\nu M \frac{d^3\psi}{d\xi^3} - \omega^2\psi \pm i\nu\omega\nu\psi = 0 \quad (22)$$

together with the following boundary conditions:

$$\begin{aligned} \psi = \frac{d\psi}{d\xi} = 0, \quad \text{at } \xi = 0 \\ \frac{d^2\psi}{d\xi^2} \pm i\nu\omega\delta \frac{d^2\psi}{d\xi^2} - i\nu M(1 - \kappa) \frac{d\psi}{d\xi} = 0, \quad \text{at } \xi = 1 \\ \frac{d^3\psi}{d\xi^3} \pm i\nu\omega\delta \frac{d^3\psi}{d\xi^3} - i\nu M \frac{d^2\psi}{d\xi^2} = 0, \quad \text{at } \xi = 1 \end{aligned} \quad (23)$$

Eq. (22) is a fourth order ordinary, linear, homogeneous, differential equation with constant complex coefficients. Therefore, instead of writing a general solution for ψ in terms of the coefficients of different terms in (22), it may also be written as

$$\psi = \sum_{j=1}^4 A_j e^{\bar{\lambda}_j \xi} \quad (24)$$

where A_j are arbitrary constants and $\bar{\lambda}_j$ are the roots of the following characteristic equation:

$$(1 \pm i\nu\omega\delta)\bar{\lambda}^4 - i\nu M\bar{\lambda}^3 - (\omega^2 \mp i\nu\omega\nu) = 0 \quad (25)$$

Substitution of (24) into (23) results in the following system of linear homogeneous equations in A_j :

$$\sum_{j=1}^4 A_j = 0 \quad (26a)$$

$$\sum_{j=1}^4 \bar{\lambda}_j A_j = 0 \quad (26b)$$

$$\sum_{j=1}^4 \left\{ (1 \pm i\nu\omega\delta)\bar{\lambda}_j^2 - i\nu M(1 - \kappa)\bar{\lambda}_j \right\} e^{\bar{\lambda}_j} A_j = 0 \quad (26c)$$

$$\sum_{j=1}^4 \left\{ (1 \pm i\nu\omega\delta)\bar{\lambda}_j^3 - i\nu M\bar{\lambda}_j^2 \right\} e^{\bar{\lambda}_j} A_j = 0 \quad (26d)$$

For a nontrivial solution, the determinant of the coefficients of A_j in (26) should be zero, which is represented as

$$\bar{\Delta} = \bar{\Delta}(\bar{\lambda}_j, (1 \pm i\nu\omega\delta), i\nu M(1 - \kappa)) = 0 \quad (27)$$

In order to expand (27) in a power series of ν , the coefficients of which are obtainable from the frequency equation of free oscillation expressed as $\Delta(\lambda_j) = 0$, let

$$\bar{\lambda}_j = \lambda_j + i\nu a_j + O(\nu^2) \quad (28)$$

which results in

$$a_j = - \frac{\omega\delta\lambda_j^4 - M\lambda_j^3 + \omega\gamma}{4\lambda_j^3}; \quad j = 1, 2, \dots, 4 \quad (29)$$

$$\lambda_{1,2} = \pm (\omega)^{\frac{1}{2}}; \quad \lambda_{3,4} = \pm i(\omega)^{\frac{1}{2}} \quad (30)$$

The determinant $\bar{\Delta}$, when expanded in powers of ν , is expressed as

$$\bar{\Delta} = \Delta(\lambda_j) + (i\nu) \sum_{j=1}^4 \frac{\partial \Delta}{\partial \lambda_j} a_j + O(\nu^2) = 0 \quad (31)$$

where

$$\begin{aligned} \Delta(\lambda_j) = & \lambda_1^2 \lambda_2^2 (\lambda_4 - \lambda_3)(\lambda_2 - \lambda_1) e^{(\lambda_1 + \lambda_2)} + \lambda_1^2 \lambda_3^2 (\lambda_2 - \lambda_4)(\lambda_3 - \lambda_1) e^{(\lambda_1 + \lambda_3)} \\ & + \lambda_1^2 \lambda_4^2 (\lambda_3 - \lambda_2)(\lambda_4 - \lambda_1) e^{(\lambda_1 + \lambda_4)} + \lambda_2^2 \lambda_3^2 (\lambda_4 - \lambda_1)(\lambda_3 - \lambda_1) e^{(\lambda_2 + \lambda_3)} \\ & + \lambda_2^2 \lambda_4^2 (\lambda_1 - \lambda_3)(\lambda_4 - \lambda_2) e^{(\lambda_2 + \lambda_4)} + \lambda_3^2 \lambda_4^2 (\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3) e^{(\lambda_3 + \lambda_4)} \end{aligned} \quad (32)$$

After simplification of (31) and neglecting $O(v^2)$ and higher, the following expression is obtained:

$$\begin{aligned} (1 + \text{Ch } \sqrt{\omega} \cos \sqrt{\omega}) + (iv) \left\{ \frac{M}{2} \left[1 + \text{Ch } \sqrt{\omega} \cos \sqrt{\omega} + \frac{1}{\sqrt{\omega}} (\text{Sh } \sqrt{\omega} \cos \sqrt{\omega} \right. \right. \\ \left. \left. + \text{Ch } \sqrt{\omega} \sin \sqrt{\omega}) \right] \mp \frac{\omega}{4} \left(\delta + \frac{\gamma}{\omega^2} \right) \left[6(1 + \text{Ch } \sqrt{\omega} \cos \sqrt{\omega}) \right. \right. \\ \left. \left. + \sqrt{\omega} (\text{Sh } \sqrt{\omega} \cos \sqrt{\omega} - \text{Ch } \sqrt{\omega} \sin \sqrt{\omega}) \right] \right\} = 0 \end{aligned} \quad (33)$$

It is remarkable that Eq. (33) is independent of κ . In other words, the so-called "degree of nonconservativeness" κ has no effect on this system. In particular, for $\kappa = 0$ (the twisting moment is always parallel to the undeformed axis of the bar) and for $\kappa = 1$ (the twisting moment is always normal to the end cross-section), the governing stability equation is the same.

To find the roots of Eq. (33), let $\sqrt{\omega} = \alpha + iv\beta$, and obtain from (33), after neglecting $O(v^2)$ and higher orders, the following:

$$1 + \text{Ch } \alpha \cos \alpha = 0 \quad (34)$$

and

$$\beta = \pm \frac{\alpha^3}{4} \left(\delta + \frac{\gamma}{\alpha^4} \right) - \frac{M}{2\alpha} \left[\frac{\text{Sh } \alpha \cos \alpha + \text{Ch } \alpha \sin \alpha}{\text{Sh } \alpha \cos \alpha - \text{Ch } \alpha \sin \alpha} \right] \quad (35)$$

The roots of (34) are, of course, the roots that correspond to the natural frequencies of an elastic cantilevered bar. Corresponding to any of these roots, the term in brackets on the right-hand side of Eq. (35) is a negative quantity. Thus, for the case of an elastic bar without external damping ($\delta = \gamma = 0$), β is a positive quantity and, therefore, the solution $e^{-i\omega\tau}$ will denote exponentially increasing motion in time. Hence, for this case the bar is unstable for any nonvanishing magnitude of the twisting moment.

If the inequality, $-\frac{\alpha^4}{2} \left(\delta + \frac{\gamma}{\alpha^4} \right) - M \left[\frac{\text{Sh } \alpha \cos \alpha + \text{Ch } \alpha \sin \alpha}{\text{Sh } \alpha \cos \alpha - \text{Ch } \alpha \sin \alpha} \right] < 0$,

holds, then the system is stable. Therefore, the critical moment is obtained by setting

$$-\frac{\alpha^4}{2} \left(\delta + \frac{\gamma}{\alpha^4} \right) - M \left[\frac{\text{Sh } \alpha \cos \alpha + \text{Ch } \alpha \sin \alpha}{\text{Sh } \alpha \cos \alpha - \text{Ch } \alpha \sin \alpha} \right] = 0$$

which yields

$$M = - \frac{(\delta \alpha^4 + \gamma)(\text{Sh } \alpha \cos \alpha - \text{Ch } \alpha \sin \alpha)}{2(\text{Sh } \alpha \cos \alpha + \text{Ch } \alpha \sin \alpha)}$$

For the smallest root of (34), $\alpha = 1.875$, the above gives

$$M = 11.54\delta + 0.925\gamma.$$

Thus it is seen that the critical moment depends linearly on both the internal and external damping.

References

1. Ziegler, H., "Die Stabilitätskriterien der Elastomechanik," Ingenieur-Archiv., Vol. 20, 1952, pp. 49-56.
2. Ziegler, H., "Linear Elastic Stability," ZAMP, Vol. 4, 1953, pp. 89-121, 168-185; see also Ziegler, H., "On the Concept of Elastic Stability," Advances in Appl. Mech., Vol. 4, edited by H. L. Dryden and T. von Karman, Academic Press, Inc., New York, N.Y., 1956, pp. 351-403.
3. Bolotin, V. V., Nonconservative Problems of the Theory of Elastic Stability, Moscow, 1961; English translation published by Pergamon Press, Inc., New York, N.Y., 1963.
4. Nemat-Nasser, S., Prasad, S. N., and Herrmann, G., "Destabilizing Effect of Velocity-Dependent Forces in Nonconservative Continuous Systems," AIAA Journal, Vol. 4, No. 7, August 1966, pp. 1276-1280.
5. Bisplinghoff, R. L., Ashley, H., and Halfman, R. L., Aeroelasticity, Addison-Wesley Publishing Co., Inc., Cambridge, Massachusetts, 1955.
6. Fung, Y. C., The Theory of Aeroelasticity, John Wiley & Sons, Inc., New York, N.Y., 1955.
7. Henry, C. J., Dugundji, J., and Ashley, H., "Aeroelastic Stability of Lifting Surfaces in High-Density Fluids," J. of Ship Research, Vol. 2, No. 4, March 1959, pp. 10-21.
8. Dugundji, J., and Ghareeb, N., "Pure Bending Flutter of a Swept Wing in a High-Density, Low-Speed Flow," AIAA Journal, Vol. 3, No. 6, June 1965, pp. 1126-1133.

9. Prasad, S. N., Nemat-Nasser, S., and Herrmann, G., "Bending-Torsional Flutter of a Swept Wing in a High-Density, Low-Speed Flow," AIAA Paper No. 66-475, to be published in AIAA Journal.
10. Bland, D. R., The Theory of Linear Viscoelasticity, Pergamon Press, Inc., New York, N.Y., 1960.
11. Nikolai, E. L., "On the Stability of the Rectilinear Form of Equilibrium of a Bar in Compression and Torsion," Izv. Leningr. politekhn. in-ta, Vol. 31, 1928; see also Nikolai, E. L., (Trudy po mekhanike), Gostekhizdat, 1955, p. 357.
12. Herrmann, G., and Jong, I. C., "On the Destabilizing Effect of Damping in Nonconservative Elastic Systems," J. of Appl. Mech., Trans. ASME, Vol. 32, No. 3, Sept. 1965, pp. 492-497.
13. Herrmann, G., and Jong, I. C., "On Nonconservative Stability Problems of Elastic Systems with Slight Damping," J. of Appl. Mech., Trans. ASME, Vol. 33, No. 1, March 1966, pp. 125-133.
14. Nemat-Nasser, S., and Herrmann, G., "Some General Considerations concerning the Destabilizing Effect in Nonconservative Systems," ZAMP, Vol. 17, 1966, pp. 305-313.
15. Herrmann, G., and Nemat-Nasser, S., "Instability Modes of Cantilevered Bars Induced by Fluid Flow through Attached Pipes," Int. J. of Solids & Structures, to be published.
16. Leipholz, H., "Über den Einfluss der Dämpfung bei nichtkonservativen Stabilitätsproblemen elastischer Stäbe," Ingenieur-Archiv., Vol. 33, 1964, pp. 308-321.

17. Trösch, A., "Stabilitätsprobleme bei tordierten Stäben und Wellen,"
Ingenieur-Archiv., Vol. 20, No. 4, 1952, pp. 258-277.

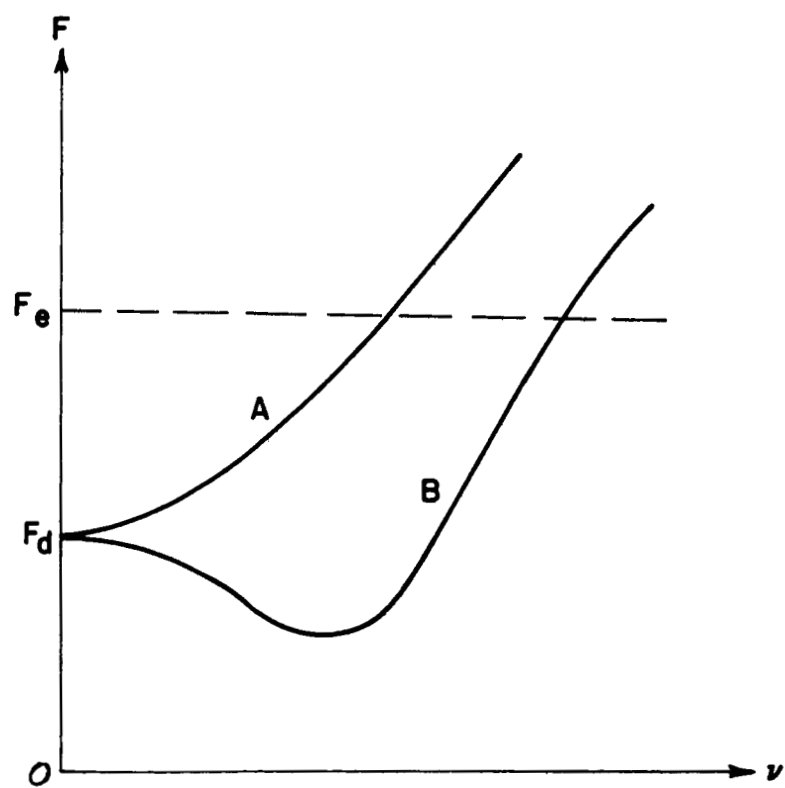


Fig. 1. Stability curves indicating dependence on ν

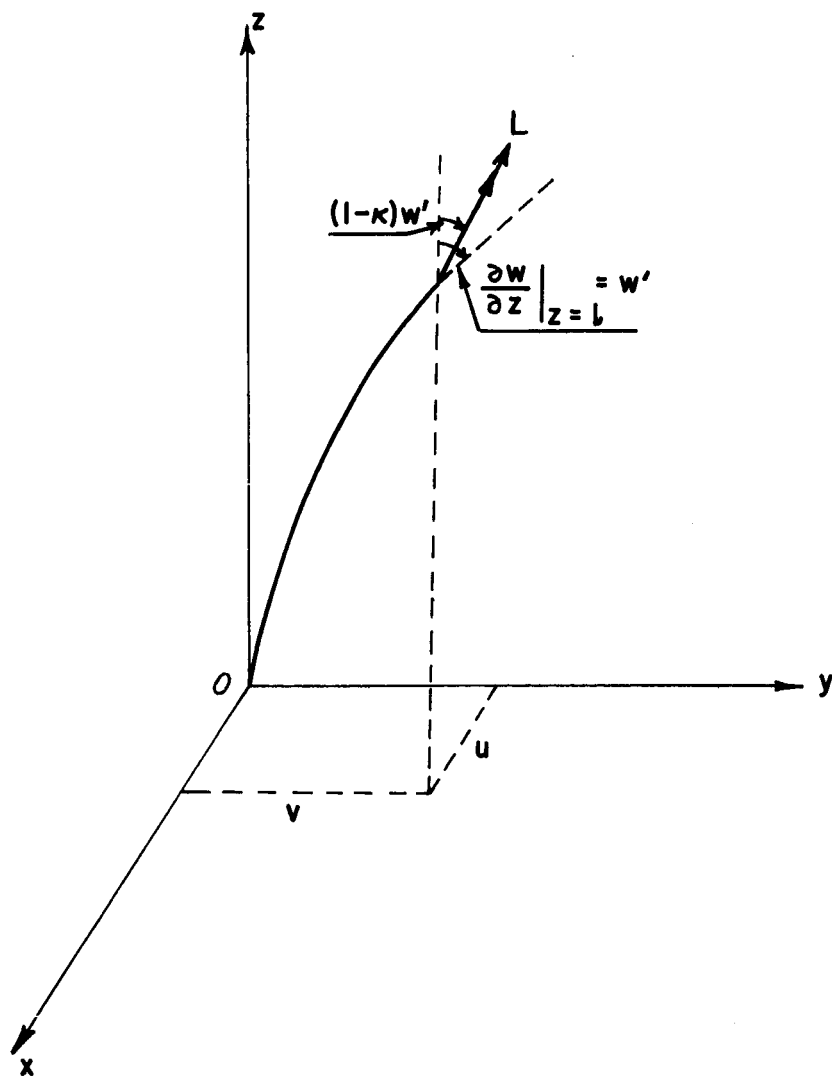


Fig. 2. Cantilevered bar subjected to a partially follower couple: geometry and layout